Qualitative properties for a class of non-autonomous semi-linear 3^{rd} order PDE arising in dissipative problems*

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Abstract

We improve results [6, 3, 4, 5] regarding the stability and attractivity of solutions u of a large class of initial-boundary-value problems of the form

$$\begin{cases} -\varepsilon(t)\, u_{xxt} + u_{tt} - C(t)\, u_{xx} + (a'{+}a)u_t = F(u), \qquad x\in]0,\pi[,\ \ t>t_0, \\ u(0,t) = 0, \quad u(\pi,t) = 0, \end{cases} \tag{1}$$

$$u(x,t_0) = u_0(x),\ \ u_t(x,t_0) = u_1(x), \quad \text{with } u_0(0) = u_1(0) = u_0(\pi) = u_1(\pi) = 0. \tag{2}$$

Here $t_0 \geq 0$, $\varepsilon \in C^2(I,I)$, $C \in C^1(I,\mathbb{R}^+)$ (with $I := [0,\infty[$) are functions of t, with $C(t) \geq \overline{C} = \text{const} > 0$; $F(0) = 0$, so that (1) admits the null solution $u^0(x,t) \equiv 0$; $a' = \text{const} \geq 0$, $a = a(x,t,u,u_x,u_t,u_{xx}) \geq 0$, $\varepsilon(t) \geq 0$. In the proof we use Liapunov functionals W depending on two parameters, which we adapt to the 'error' σ.

Key Words: *Nonlinear higher order PDE, Stability, Boundary value problems*

1 Introduction

The class (1-2) includes (see e.g. the introduction of [6]) equations arising in Superconductor Theory [8, 1, 2] and in the Theory of Viscoelastic Materials [9]. We generalize theorem 3.1 of [6], to which we refer also for examples. To formulate the notions of stability and attractivity[10, 7] we use the distance $d(t) := d(u, u_t, t)$ between u, u^0, where the norm $d(\varphi, \psi, t)$ is defined by

$$d^2(\varphi, \psi, t) := \int_0^\pi [\varepsilon^2(t)\varphi_{xx}^2 + \varphi_x^2 + \varphi^2 + \psi^2] dx. \tag{3}$$

ε^2 plays the role of a t-dependent weight for φ_{xx}^2; for $\varepsilon \equiv 0$, d reduces to the norm needed for the corresponding second order problem. The vanishing of φ, ψ in $0, \pi$ implies $|\varphi(x)|, \varepsilon(t)|\varphi_x(x)| \leq d(\varphi, \psi, t)$ for all x; a convergence w.r.t. d therefore implies a uniform (in x) pointwise convergence

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of φ, and also of φ_x if $\varepsilon(t)\neq 0$. Throughout the paper $t_0\in I_\kappa:=[\kappa,\infty[$, $\kappa\in\underline{\mathrm{R}}$, $\xi>0$. For any function $f(t)$ we denote $\overline{f}=\inf_{t>0} f(t)$, $\overline{\overline{f}}=\sup_{t>0} f(t)$.

Def. 1.1 u^0 is *stable* if for any $\sigma\in]0,\xi]$ there exists a $\delta(\sigma,t_0)>0$ such that

$$d(t_0)<\delta(\sigma,t_0) \qquad\Rightarrow\qquad d(t)<\sigma \quad \forall t\geq t_0\in I_\kappa. \tag{4}$$

u^0 is *uniformly stable* if δ can be chosen independent of t_0, $\delta=\delta(\sigma)$.

Def. 1.2 u^0 is *asymptotically stable* if it is stable and $\forall t_0\in I_\kappa$, $\nu>0$ there exist $\delta(t_0)>0$, $T(\nu,t_0,u_0,u_1)>0$ such that:

$$d(t_0)<\delta \qquad\Rightarrow\qquad d(t)<\nu \quad \forall t\geq t_0+T. \tag{5}$$

Def. 1.3 u^0 is *uniformly exponential-asymptotically stable* if $\exists\delta,D,E>0$:

$$d(t_0)<\delta \qquad\Rightarrow\qquad d(t)\leq D\exp\left[-E(t-t_0)\right]d(t_0), \quad \forall t\geq t_0\in I_\kappa. \tag{6}$$

2 Main assumptions and preliminary estimates

Assumptions I: We assume that there exist constants $k\geq 0$, $h\geq 0$, $A\geq 0$, $\omega>0$, $\rho>0$, $\mu>0$, $\tau>0$ such that

$$F(0)=0, \qquad F_z(z)\leq k+h|z|^\omega \qquad \text{if } |z|<\rho. \tag{7}$$

$$\overline{C}>k, \qquad C-\dot{\varepsilon}\geq\mu(1+\varepsilon), \qquad \mu+\overline{C}/2-2k>0, \qquad \overline{\overline{\dot{\varepsilon}}}>-\infty. \tag{8}$$

$$0\leq a\leq Ad^\tau(u,u_t,t), \qquad a'+\overline{\varepsilon}/2>0 \tag{9}$$

(we are not excluding $a'<0$). Setting $h=0$ in (7) one obtains the analog assumption considered in Ref.[6] ; the present one is slightly more general as it may be satisfied with a smaller k, what makes $(8)_1$ weaker, and/or a larger ρ. Upon integration (7) implies for all $|\varphi|<\rho$

$$\varphi F(\varphi)\leq k\varphi^2+\tfrac{h}{\omega+1}|\varphi|^{\omega+2}, \qquad \int_0^\varphi F(z)dz\leq k\tfrac{\varphi^2}{2}+\tfrac{h|\varphi|^{\omega+2}}{(\omega+1)(\omega+2)}. \tag{10}$$

We recall Poincaré inequality, which easily follows from Fourier analysis:

$$\phi\in C^1(]0,\pi[),\;\; \phi(0)=0,\;\; \phi(\pi)=0, \qquad\Rightarrow\qquad \int_0^\pi\phi_x^2(x)dx\geq\int_0^\pi\phi^2(x)dx. \tag{11}$$

We introduce the non-autonomous **family of Liapunov functionals**[6]

$$W(\varphi,\psi,t;\gamma,\theta)=\int_0^\pi\left[\gamma\psi^2+(\varepsilon\varphi_{xx}-\psi)^2+[C(1+\gamma)+\varepsilon(a'+\theta)-\dot{\varepsilon}]\varphi_x^2+a'\theta\varphi^2+2\theta\varphi\psi-2(1+\gamma)\int_0^{\varphi(x)}F(z)dz\right]\frac{dx}{2}$$

depending on two for the moment unspecified positive parameters θ,γ. Let $W(t;\gamma,\theta):=W(u,u_t,t;\gamma,\theta)$. In Ref. [6] we have found

$$\dot W=-\int\limits_0^\pi\!\Big\{\varepsilon\gamma u_{xt}^2+\Big[(a{+}a')(1{+}\gamma)-\theta-\frac{\varepsilon a^2}{C-\dot\varepsilon}-\frac{\theta a^2}{C}\Big]u_t^2+\varepsilon(C{-}\dot\varepsilon)\Big[\frac{au_t}{C-\dot\varepsilon}-\frac{u_{xx}}{2}\Big]^2+\frac{3\varepsilon}{4}(C-\dot\varepsilon)u_{xx}^2$$
$$+\Big[C\big(\tfrac{\theta}{2}-a'\big)+\ddot\varepsilon+(C{-}\dot\varepsilon)(a'{+}\theta)-(1{+}\gamma)\dot C-2\varepsilon F_u\Big]\tfrac{u_x^2}{2}+\tfrac{\theta C}{4}(u_x^2-u^2)+\tfrac{\theta C}{4}\big[u+\tfrac{2a}{C}u_t\big]^2-\theta uF\Big\}dx$$

Provided $|u|<\rho$, $\theta>\max\{2a',-a'\}$, $\mu(a'+\theta)>2k$, (11) with $\phi=u_t,u$, implies

$$\begin{aligned}\dot W\le&-\int_0^\pi\!\Big\{\Big[\overline{\varepsilon}\gamma+(a+a')(1+\gamma)-\theta-a^2\Big(\frac{1}{\mu}+\frac{\theta}{\overline{\overline{C}}}\Big)\Big]u_t^2+\frac{3}{4}\mu\varepsilon^2u_{xx}^2+\Big[\overline{C}\Big(\frac{\theta}{2}-a'\Big)+\overline{\ddot\varepsilon}\\
&+\mu(a'+\theta)+[\mu(a'+\theta)-2(k+h|u|^\omega)]\varepsilon-(1+\gamma)\dot C\Big]\tfrac{u_x^2}{2}-\theta\Big(ku^2+\tfrac{h}{\omega+1}|u|^{\omega+2}\Big)\Big\}dx\\
\le&-\int_0^\pi\!\Big\{\Big[\overline{\varepsilon}\gamma+(a+a')(1+\gamma)-\theta-a^2\Big(\frac{1}{\mu}+\frac{\theta}{\overline{C}}\Big)\Big]u_t^2+\frac{3\mu}{4}\varepsilon^2u_{xx}^2+\Big[\theta\Big(\mu+\frac{\overline{C}}{2}-2k\Big)+\overline{\ddot\varepsilon}\\
&-(1+\gamma)\dot C+a'(\mu-\overline{C})+[\mu(a'+\theta)-2k]\varepsilon\Big]\tfrac{u_x^2}{2}-h\varepsilon|u|^\omega u_x^2-\tfrac{h\theta}{\omega+1}|u|^{\omega+2}\Big\}dx.\end{aligned}\tag{12}$$

To find an upper bound for $\dot W$ we make **Assumption II**:

$$\forall\gamma>0\quad\exists\bar t(\gamma)\in[0,\infty[\quad\text{such that }\dot C(1+\gamma)\le 1\quad\text{for }t\ge\bar t.\tag{13}$$

(13) is fulfilled by $\bar t(\gamma)\equiv 0$ if $\dot C\le 0$, by some $\bar t(\gamma)\ge 0$ if $\dot C\overset{t\to\infty}{\longrightarrow}0$. (13) implies $\overline{\ddot\varepsilon}\le 0$: $\overline{\ddot\varepsilon}>0$ would imply $\dot\varepsilon\ge\overline{\ddot\varepsilon}t+\dot\varepsilon(0)$, $\varepsilon\ge\overline{\ddot\varepsilon}t^2/2+\dot\varepsilon(0)t+\varepsilon(0)$ and by $(8)_2$ that C grows at least quadratically with t, against (13). We choose

$$\begin{aligned}&\theta>\theta_1:=\max\Big\{2a',\,\tfrac{2k}{\mu}-a',\,\tfrac{5-\overline{\ddot\varepsilon}-a'(\mu-\overline{C})}{\mu+\overline{C}/2-2k}\Big\},\\
&\gamma>\gamma_1(\sigma):=\tfrac{1+\theta+\overline{\varepsilon}/2}{a'+\overline{\varepsilon}}+\gamma_{32}\sigma^{2\tau}\qquad\gamma_{32}:=\tfrac{A^2}{(a'+\overline{\varepsilon})}\Big(\tfrac{1}{\mu}+\tfrac{\theta}{\overline{C}}\Big).\end{aligned}\tag{14}$$

These definitions respectively imply, provided $t>\bar t$ and $d(t)\le\sigma<\rho$,

$$\begin{aligned}&\theta\big(\mu+\overline{C}/2-2k\big)+[\mu(a'+\theta)-2k]\overline{\varepsilon}+\overline{\ddot\varepsilon}-(1+\gamma)\dot C+a'(\mu-\overline{C})>4,\\
&\overline{\varepsilon}\gamma+(a+a')(1+\gamma)-\theta-a^2\Big(\tfrac{1}{\mu}+\tfrac{\theta}{\overline{C}}\Big)\ge a'+\tfrac{a+a'+\overline{\varepsilon}}{a'+\overline{\varepsilon}}\,[(1+\theta+\overline{\varepsilon}/2)\\
&\;+A^2\Big(\tfrac{1}{\mu}+\tfrac{\theta}{\overline{C}}\Big)\sigma^{2\tau}\Big]-\theta-A^2\Big(\tfrac{1}{\mu}+\tfrac{\theta}{\overline{C}}\Big)d^{2\tau}\ge 1+a'+\overline{\varepsilon}/2>1.\end{aligned}\tag{15}$$

If $0<d(t)<\sigma$ (12), (15) imply for all $t\ge\bar t$ the **upper bound for $\dot W$**

$$\begin{aligned}\dot W(u,u_t,t;\gamma,\theta)&\le-\eta\,d^2(t)+\int\limits_0^\pi h\Big[\varepsilon|u|^\omega u_x^2+\frac{\theta}{\omega+1}|u|^{\omega+2}\Big]dx\\
&\le\Big[-\eta+h2^{\frac{\omega}{2}}\Big(\varepsilon(t)+\tfrac{\theta}{\omega+1}\Big)d^\omega(t)\Big]d^2(t),\qquad\eta:=\min\big\{1,\tfrac{3}{4}\mu\big\}.\end{aligned}\tag{16}$$

From the definition of W it immediately follows

$$W(\varphi,\psi,t;\gamma,\theta)=\int_0^{\pi}\frac{1}{2}\Big\{\Big(\gamma-\theta^2-\frac{1}{2}\Big)\,\psi^2+\frac{(\varepsilon\varphi_{xx}-2\psi)^2}{4}+\frac{(\varepsilon\varphi_{xx}-\psi)^2}{2}+\varepsilon^2\frac{\varphi_{xx}^2}{4}$$
$$+[C(1+\gamma)-\dot{\varepsilon}+\varepsilon(a'+\theta)]\varphi_x^2+(\theta a'-1)\varphi^2+[\theta\psi+\varphi]^2-2(1+\gamma)\int_0^{\varphi(x)}F(z)dz\Big\}\,dx.$$

Using $(8)_2$, (10) and (11) with $\phi(x)=\varphi(x)$ we find for $|\varphi|<\rho$

$$W\geq\int_0^{\pi}\frac{dx}{2}\Big\{\Big[\gamma-\theta^2-\frac{1}{2}\Big]\psi^2+\frac{\varepsilon^2\varphi_{xx}^2}{4}+\Big[\mu+\Big(\mu+a'+\frac{\theta}{2}\Big)\overline{\varepsilon}\Big]\varphi_x^2+\Big[\Big(a'+\frac{\overline{\varepsilon}}{2}\Big)\theta-1-k+(\overline{C}-k)\gamma-\frac{2h(1+\gamma)|\varphi|^{\omega}}{(\omega+1)(\omega+2)}\Big]\varphi^2\Big\}\,.$$

Choosing $\theta>\theta_2:=\max\Big\{\theta_1,\frac{\overline{C}+5/4}{a'+\overline{\varepsilon}/2}\Big\}$, $\gamma\geq\gamma_2(\sigma):=\gamma_1(\sigma)+\theta^2+1$ we find

$$W>\int_0^{\pi}\frac{1}{2}\Big\{\Big[\gamma-\theta^2-\frac{1}{2}\Big]\,\psi^2+\varepsilon^2\frac{\varphi_{xx}^2}{4}+\Big[\mu+\Big(\mu+a'+\frac{\theta}{2}\Big)\,\overline{\varepsilon}\Big]\,\varphi_x^2$$
$$+\Big[\tfrac{1}{4}+(1+\gamma)\Big(\overline{C}-k-\tfrac{2h|\varphi|^{\omega}}{(\omega+1)(\omega+2)}\Big)\Big]\,\varphi^2\Big\}\,dx.$$

By the inequality $|\varphi|<d$ the expression in the last bracket is positive if

$$d(t)\leq\sigma<\rho_2:=\min\Big\{\rho,\,\big[(\overline{C}-k)(\omega+1)(\omega+2)/2h\big]^{1/\omega}\Big\}\,.$$

Hence for $d\leq\sigma$ the last square bracket is larger than $1/4$, and we find the **lower bound for** W

$$W(\varphi,\psi,t;\gamma,\theta)\geq\chi d^2(\varphi,\psi,t),\qquad\chi:=\tfrac{1}{2}\min\{\tfrac{1}{4},\mu+(\mu+a'+\tfrac{\theta}{2})\overline{\varepsilon}\}>0.\tag{17}$$

We also recall the **upper bound for** W proved in [6] for $d\leq\sigma$:

$$W(\varphi,\psi,t;\gamma,\theta)\leq[1+\gamma(\sigma)]\,g(t)B^2(d).\tag{18}$$

The map $d\in[0,\infty[\to B(d)\in[0,\infty[$ is continuous and increasing, hence invertible. Moreover, $B(d)\geq d$. Here we have chosen γ and defined

$$\begin{aligned}
&\gamma\geq\gamma_3(\sigma):=\gamma_2(\sigma)+1+\tfrac{a'+\theta}{\mu}+(a'+1)\theta=\gamma_{31}+\gamma_{32}\sigma^{2\tau},\\
&\gamma_{31}:=\tfrac{1+\theta}{a'+\overline{\varepsilon}}+\theta^2+2+\tfrac{a'+\theta}{\mu}+(a'+1)\theta,\qquad g(t):=C(t)-\tfrac{\dot{\varepsilon}(t)}{2}+1>1,\\
&m(r):=\max\{|F_\zeta(\zeta)|\;:\;|\zeta|\leq r\}\,,\qquad B^2(d):=[1+m(d)]\,d^2.
\end{aligned}\tag{19}$$

Fixed $\sigma\in]0,\rho_2[$, if $d<\sigma$ we find $B^2(d)\leq[1+m(\sigma)]d^2$ and, by (16-18),

$$\begin{aligned}
&\dot{W}<-lW+nW^{1+\frac{\omega}{2}},\\
&n(t):=\tfrac{h2^{\frac{\omega}{2}}}{\chi^{1+\frac{\omega}{2}}}\Big[\tfrac{\theta}{\omega+1}+\varepsilon(t)\Big],\qquad l(t,\sigma):=\tfrac{\lambda(\sigma)}{g(t)},\qquad\lambda(\sigma):=\tfrac{\eta}{[1+m(\sigma)][1+\gamma_3(\sigma)]}.
\end{aligned}\tag{20}$$

$\lambda(\sigma)$ is positive-definite and decreasing. By the Comparison Principle [10], $W(t)<y(t)$ for $t>t_0$, where $y(t)$ solves the Cauchy problem

$$\dot{y}=-ly+ny^{1+\omega/2}, \qquad\qquad y(t_0)=W_0:=W(t_0)$$

and we have to choose $t_0\geq\bar{t}$. As known, the change of variable $z=y^{-\omega/2}$ reduces this Bernoulli equation to the linear one $\dot{z}=zl\omega/2-n\omega/2$, which is easily solved to give the following **comparison equation for** W for $t>t_0$:

$$W(t)<y(t)=W_0\,e^{-\lambda\int\limits_{t_0}^{t}\frac{d\tau}{g(\tau)}}\left\{1-W_0^{\frac{\omega}{2}}\frac{\omega}{2}\int\limits_{t_0}^{t}n(\tau)e^{-\frac{\omega\lambda}{2}\int_{t_0}^{\tau}\frac{d\tau'}{g(\tau')}}d\tau\right\}^{-\frac{2}{\omega}} \tag{21}$$

A sufficient condition for $\dot{W}(t)$ to be negative is that $n/l<W^{-\frac{\omega}{2}}$, namely

$$\frac{n(t)g(t)}{\lambda}<W_0^{-\frac{\omega}{2}}e^{\frac{\omega\lambda}{2}\int_{t_0}^{t}\frac{d\tau}{g(\tau)}}\left\{1-W_0^{\frac{\omega}{2}}\frac{\omega}{2}\int_{t_0}^{t}n(\tau)e^{-\frac{\omega\lambda}{2}\int_{t_0}^{\tau}\frac{d\tau'}{g(\tau')}}d\tau\right\},$$

or equivalently, after some algebra, that

$$\begin{aligned}&W_0^{-\frac{\omega}{2}}>s(t;t_0,\sigma),\\&s(t;t_0,\sigma):=\frac{n(t)g(t)}{\lambda(\sigma)}e^{-\frac{\omega\lambda(\sigma)}{2}\int_{t_0}^{t}\frac{d\tau}{g(\tau)}}+\frac{\omega}{2}\int_{t_0}^{t}n(\tau)e^{-\frac{\omega\lambda(\sigma)}{2}\int_{t_0}^{\tau}\frac{d\tau'}{g(\tau')}}d\tau.\end{aligned} \tag{22}$$

Summing up, $W(t)$ is decreasing and fulfills (21) in $[t_0,\infty[$ if $d(t)<\sigma$ and (22) is satisfied for all $t\geq t_0$, or equivalently if

$$S(t_0,\sigma):=\sup_{[t_0,\infty[}s(t;t_0,\sigma)<\infty, \qquad \Delta(t_0,\sigma):=S(t_0,\sigma)W_0^{\frac{\omega}{2}}<1. \tag{23}$$

We give upper bounds for $s(t;t_0,\sigma)$, $S(t_0,\sigma)$ using g only: $(19)_3$, $(8)_2$ imply

$$g=\tfrac{1}{2}[C-\dot{\varepsilon}]+\tfrac{C}{2}+1\geq\tfrac{\mu}{2}(1+\varepsilon)+\tfrac{C}{2}+1 \qquad\Rightarrow\qquad 0\leq n(t)\leq\alpha_1[\alpha_2+g(t)],$$

where $\alpha_1=\frac{h2^{1+\frac{\omega}{2}}}{\mu\chi^{1+\frac{\omega}{2}}}$, $\alpha_2=\left[\frac{\mu\theta}{\omega+1}-\mu-2-\overline{C}\right]/2$. Hence, as announced,

$$\begin{aligned}s(t;t_0,\sigma)&\leq\tfrac{\alpha_1}{\lambda}[\alpha_2+g(t)]g(t)e^{-\frac{\omega\lambda}{2}\int\limits_{t_0}^{t}\frac{d\tau}{g(\tau)}}+\tfrac{\omega}{2}\int\limits_{t_0}^{t}\alpha_1[\alpha_2+g(\tau)]e^{-\frac{\omega\lambda}{2}\int\limits_{t_0}^{\tau}\frac{d\tau'}{g(\tau')}}d\tau\\&=\tfrac{\alpha_1}{\lambda}[\alpha_2+g(t_0)]g(t_0)+\tfrac{\alpha_1}{\lambda}\int_{t_0}^{t}e^{-\frac{\omega\lambda}{2}\int_{t_0}^{\tau}\frac{d\tau'}{g(\tau')}}\dot{g}(\tau)[\alpha_2+2g(\tau)]d\tau\\&\leq\tfrac{\alpha_1}{\lambda}[\alpha_2+g(t_0)]g(t_0)+\tfrac{\alpha_1}{\lambda}\left[\tfrac{1}{1+\gamma_3(\sigma)}-\tfrac{\overline{\ddot{\varepsilon}}}{2}\right]\int_{t_0}^{t}e^{-\frac{\omega\lambda}{2}\int_{t_0}^{\tau}\frac{d\tau'}{g(\tau')}}[\alpha_2+2g(\tau)]d\tau\end{aligned} \tag{24}$$

where we have integrated by parts and used (13) to get $\dot{g}=\dot{C}-\ddot{\varepsilon}/2\leq 1/(1+\gamma_3)-\overline{\ddot{\varepsilon}}/2$. As $\overline{\ddot{\varepsilon}}\leq 0$, the second square bracket is positive; the last integral is an increasing function of t as its argument is positive, whence

$$S(t_0,\sigma)\leq\tfrac{\alpha_1}{\lambda}[\alpha_2+g(t_0)]g(t_0)+\tfrac{\alpha_1}{\lambda}\left[\tfrac{1}{1+\gamma_3(\sigma)}-\tfrac{\overline{\ddot{\varepsilon}}}{2}\right]\int\limits_{t_0}^{\infty}e^{-\frac{\omega\lambda}{2}\int_{t_0}^{\tau}\frac{d\tau'}{g(\tau')}}[\alpha_2+2g(\tau)]d\tau,$$

and $S(t_0,\sigma)<\infty$ for all $t_0\ge 0$ if

$$G(\sigma):=h\int_0^\infty e^{-\frac{\omega\lambda(\sigma)}{2}\int_0^\tau\frac{d\tau'}{g(\tau')}}g(\tau)d\tau<\infty. \tag{25}$$

Let $\sigma'_M:=\sup\{\sigma\in\mathbb{R}^+|\,G(\sigma)<\infty\}$. If $h=0$, then $G(\sigma)\equiv 0$, $\sigma'_M=\infty$ and any W_0 fulfills $(23)_2$. It is $\sigma'_M=\infty$ also if $h>0$ and e.g. $g(t)\le K'+K''t^a$ with some $K',K''>0$, $0\le a<1$; whereas $h>0$ and e.g. $g(t)\le K'+Kt$ with some $K'>0$, $K\in]0,\frac{\omega\lambda(\sigma)}{4}[$ gives a finite $\sigma'_M>0$, determined by $\lambda(\sigma'_M)=4K/\omega$.

The inequality $\sigma'_M>0$ and (25) imply $\int_0^\infty\frac{dt}{g(t)}=\infty$: in fact, if it were $\int_0^\infty\frac{dt}{g(t)}<\infty$ it would be $e^{-\frac{\omega\lambda(\sigma)}{2}\int_0^\tau\frac{d\tau'}{g(\tau')}}>L:=e^{-\frac{\omega\lambda(\sigma)}{2}\int_0^\infty\frac{d\tau'}{g(\tau')}}>0$, whence $G(\sigma)>hL\int_0^\infty g(\tau)d\tau=\infty$, for *all* $\sigma>0$.

3 Stability and asymptotic stability of the null solution u^0

Theorem 3.1 *Assume conditions (7-9) and either $\dot C\le 0$ for all $t\in I$, or $\dot C\stackrel{t\to\infty}{\longrightarrow}0$. u^0 is stable if $\sigma'_M>0$, asymptotically stable if moreover $\int_0^\infty\frac{dt}{g(t)}=\infty$. u^0 is uniformly stable and exponential-asymptotically stable if $\overline{\overline{g}}<\infty$.*

Proof. We first analyze the behaviour of $r^2(\sigma):=\frac{\sigma^2}{1+\gamma_3(\sigma)}=\frac{\sigma^2}{1+\gamma_{31}+\gamma_{32}\sigma^{2\tau}}$. By $(19)_1$ the positive constants γ_{31},γ_{32} are independent of σ,t_0. $r(\sigma)$ is an increasing and therefore invertible map $r:[0,\sigma_M[\to[0,r_M[$, where:

$$\begin{array}{lll}\sigma_M=\infty, & r_M=\infty, & \text{if }\ \tau\in[0,1[,\\ \sigma_M=\infty & r_M=1/\sqrt{\gamma_{32}}, & \text{if }\ \tau=1,\\ \sigma_M^{2\tau}:=\frac{1+\gamma_{31}}{\gamma_{32}(\tau-1)}, & r_M=[\frac{\tau-1}{1+\gamma_{31}}]^{\frac{\tau-1}{2\tau}}/\sqrt{\tau}\gamma_{32}^{\frac{1}{2\tau}}, & \text{if }\ \tau>1,\end{array}$$

[in the latter case $r(\sigma)$ is decreasing beyond σ_M]. Next, let $\xi:=\min\{\rho,\sigma_M,\sigma'_M\}$ if the rhs is finite, otherwise choose $\xi\in\mathbb{R}^+$; we shall consider an "error" $\sigma\in]0,\xi[$. We define $\kappa:=\bar t[\gamma_3(\xi)]$ and

$$\delta(\sigma,t_0):=\min\left\{B^{-1}\left[\frac{\sigma\sqrt{\chi}}{\sqrt{g(t_0)(1+\gamma_3(\sigma))}}\right],B^{-1}\left[\frac{[S(t_0,\sigma)]^{-\frac{1}{\omega}}}{\sqrt{g(t_0)(1+\gamma_3(\sigma))}}\right]\right\}. \tag{26}$$

$\delta(\sigma,t_0)$ belongs to $]0,\sigma[$, because $d\le B(d)$ implies $B^{-1}(d)\le d$, whence $B^{-1}\left[\sigma\sqrt{\chi}/\sqrt{g(t_0)(1+\gamma_3)}\right]\le\sigma/4$, and is an increasing function of σ. $\bar t(\gamma)$ was defined in (13); it is $\bar t[\gamma_3(\sigma)]\le\kappa$, as the function $\bar t[\gamma_3(\sigma)]$ is non-decreasing. Mimicking an argument of [5, 6] we show that for any $t_0\ge\kappa$, $\sigma\in]0,\xi[$

$$d(t_0)<\delta(\sigma,t_0)\qquad\Rightarrow\qquad d(t)<\sigma\qquad\forall t\ge t_0. \tag{27}$$

Ad absurdum, assume (27) is fulfilled for all $t\in[t_0,t_1[$ whereas $d(t_1)=\sigma$, with some $t_1>t_0$. (23) is trivially satisfied if $h=0$; if $h>0$ it follows from

$$W_0\le[1+\gamma_3]g(t_0)B^2\left[d(t_0)\right]<[1+\gamma_3(\sigma)]\,g(t_0)B^2\left[\delta(\sigma,t_0)\right]\le[S(t_0,\sigma)]^{-\frac{2}{\omega}},$$

where we have used (18), (26) in the first and last inequality. It implies that $W(t) \equiv W[u, u_t, t; \gamma_3(\sigma), \theta]$ is a decreasing function of t in $[t_0, t_1]$. Using (17) and again (18), (26) we find the following contradiction with $d(t_1) = \sigma$:

$$\chi d^2(t_1) \leq W(t_1) < W_0 < [1+\gamma_3(\sigma)]\, g(t_0) B^2\left[\delta(\sigma, t_0)\right] \leq \chi\sigma^2.$$

(27) amounts to the stability of u^0; if $\overline{\overline{g}}<\infty$ we can replace $g(t_0)$ by $\overline{\overline{g}}$ in the first inequality of (24) and obtain by integration the stronger inequalities

$$s(t; t_0, \sigma) \leq \tfrac{\alpha_1}{\lambda(\sigma)}\left[\alpha_2 + \overline{\overline{g}}\right]\overline{\overline{g}} \qquad \Rightarrow \qquad S(t_0, \sigma) \leq \tfrac{\alpha_1}{\lambda(\sigma)}\left[\alpha_2 + \overline{\overline{g}}\right]\overline{\overline{g}}; \tag{28}$$

because of (28) we find the uniform stability (Def. 1.1) with

$$\delta(\sigma) := \min\left\{B^{-1}\left[\frac{\sigma\sqrt{\chi}}{\sqrt{\overline{\overline{g}}(1+\gamma_3(\sigma))}}\right], B^{-1}\left[\frac{\left[\frac{\alpha_1\overline{\overline{g}}}{\lambda(\sigma)}\left(\alpha_2+\overline{\overline{g}}\right)\right]^{-\frac{1}{\omega}}}{\sqrt{\overline{\overline{g}}(1+\gamma_3(\sigma))}}\right]\right\}.$$

Let now $\delta(t_0):=\delta(\xi/2, t_0)$. By (27) we find that, for any $t_0\geq\kappa$, $d(t_0)<\delta(t_0)$ implies $d(t)<\xi/2$ for all $t\geq t_0$. Choosing $W(t)\equiv W[u,u_t,t;\gamma_3(\xi/2),\theta]$, on one hand (18) becomes $W(t)\leq\frac{\eta g(t)}{\lambda(\xi/2)}d^2(t)$, while by (22), (23)

$$W_0^{\frac{\omega}{2}} s\left(t;t_0,\tfrac{\xi}{2}\right) = W_0^{\frac{\omega}{2}}\left[\frac{n(t)g(t)}{\lambda(\xi/2)}e^{-\frac{\omega}{2}\lambda\left(\frac{\xi}{2}\right)\int_{t_0}^{t}\frac{d\tau}{g(\tau)}} + \frac{\omega}{2}\int_{t_0}^{t} n(\tau)e^{-\frac{\omega}{2}\lambda\left(\frac{\xi}{2}\right)\int_{t_0}^{\tau}\frac{d\tau'}{g(\tau')}}d\tau\right] \leq \Delta\left(t_0,\tfrac{\xi}{2}\right)$$

with $\Delta(t_0,\xi/2) < 1$, and $1 - W_0^{\frac{\omega}{2}}\frac{\omega}{2}\int_{t_0}^{t} n(\tau)e^{-\frac{\omega}{2}\lambda\left(\frac{\xi}{2}\right)\int_{t_0}^{\tau}\frac{d\tau'}{g(\tau')}}d\tau \geq 1-\Delta(t_0,\xi/2) > 0$. These inequalities and (17), (21) imply

$$d^2(t) \leq \frac{W(t)}{\chi} < \frac{W_0}{\chi}e^{-\lambda\int_{t_0}^{t}\frac{d\tau}{g(\tau)}}\left[1-\frac{\omega}{2}W_0^{\frac{\omega}{2}}\int_{t_0}^{t} n(\tau)e^{-\frac{\omega\lambda}{2}\int_{t_0}^{\tau}\frac{dz}{g(z)}}d\tau\right]^{-\frac{2}{\omega}} < \frac{\eta g(t_0)d^2(t_0)}{\lambda\chi}e^{-\lambda\int_{t_0}^{t}\frac{d\tau}{g(\tau)}}\left[1-\Delta\left(t_0,\tfrac{\xi}{2}\right)\right]^{-\frac{2}{\omega}}$$

with $\lambda=\lambda(\xi/2)$. The condition $\int_0^\infty\frac{dt}{g(t)}=\infty$ implies that the exponential goes to zero as $t\to\infty$, proving the asymptotic stability of u^0; if $\overline{\overline{g}}<\infty$ we can replace $g(t_0), g(\tau)$ by $\overline{\overline{g}}$ in the last inequality and obtain

$$d^2(t) < d^2(t_0)\frac{\eta\overline{\overline{g}}}{\lambda(\xi/2)\chi}\exp\left[-\frac{\lambda(\xi/2)}{\overline{\overline{g}}}(t-t_0)\right][-\Delta(t_0,\xi/2)]^{-\frac{2}{\omega}},$$

proving the uniform exponential-asymptotic stability of u^0: set in Def. 1.3

$$\delta=\delta\left(\xi/2, t_0\right), \qquad D=\sqrt{\frac{\eta\overline{\overline{g}}}{\lambda(\xi72)\chi}}\left[1-\Delta\left(t_0,\frac{\xi}{2}\right)\right]^{-\frac{2}{\omega}}, \qquad E=\frac{\lambda(\xi/2)}{2\overline{\overline{g}}}.$$

References

[1] A. Barone, G. Paternó *Physics and Applications of the Josephson Effect*, Wiley-Interscience, New-York, 1982; and references therein.

[2] P. l. Christiansen, A. C. Scott, M. P. Sorensen, *Nonlinear Science at the Dawn of the 21st Century*, Lecture Notes in Physics 542, Springer, 2000.

[3] B. D'Acunto, A. D'Anna, Rend. Mat. Serie **VII**, Vol. 18, (1998), 347-365.

[4] A. D'Anna, G. Fiore Rend. Mat. Serie **VII**, Vol. 21 (2000), 191-206.

[5] A. D'Anna, G. Fiore, Nonlinear Dyn. Syst. Theory **5** (2005), 9-38.

[6] A. D'Anna, G. Fiore, Nonlinear Dyn. Syst. Theory **9** (2009), 249-262.

[7] J. N. Flavin, S. Rionero, *Qualitative estimates for partial differential equations. An introduction*, CRC Press, Boca Raton, FL, 1996. 368 pp.

[8] Josephson B. D. Phys. Lett. **1** (1962), 251-253; Rev. Mod. Phys. B **46** (1974), 251-254; and references therein.

[9] P. Renno, Atti Acc. Lincei Rend. Fis. **75** (1983), 1-10.

[10] T. Yoshizawa, *Stability Theory by Liapunov's second method*, The Mathematical Society of Japan, 1966.